angle at the third vertex, and lay off twice its complement from A_4 around the circumference of the circle. This locates the desired point A_1 to complete the quadrangle.

Two special cases of quadrangles which yield four cusped hypocycloids deserve mention. We easily prove by condition (4)

THEOREM 4. *If an inscribed quadrangle is a square, or if three of its vertices form an equilateral triangle, then its Wallace lines envelop a four-cusped hypocycloid.*

Suppose we cause the point P to remian fixed while the quadrangle $A_1A_2A_3A_4$-revolves about the center of the circum-circle without changing its shape. This can be done if in equation (1) we let θ be constant and replace each θ_i by $\theta_i + \lambda$. Thus the θ_i will be initial values and λ the angle through which the quadrangle has rotated, λ being now the parameter. Then by methods similar to those we have used it is not difficult to prove

THEOREM 5. *If an inscribed quadrangle revolves about the center of its circumcircle, the Wallace line of a fixed point on the circumcircle envelops a curve whose evolute is in general a two-cusped epicycloid; but if the quadrangle is a rectangle the envelope is a circle, and if it is a square*[1] *the envelope is a point. The successive positions of the Wallace line can be described as the successive positions of a line rigidly attached to a circle which rolls upon the exterior of a fixed circle of equal radius.*

It seems a pity not to generalize some of these theorems to the case of an inscribed n-gon, but only theorem 4 seems easy to extend. Steggall has found the envelope in the case of a regular n-gon to be an n-cusped hypocycloid, which generalizes part of theorem 4; and the present writer has generalized the remainder of this theorem so that it reads:

THEOREM 6. *If an inscribed n-gon is regular, or if $n - 1$ of its vertices form a regular polygon, its Wallace lines envelop an n-cusped hypocycloid.*

GEOMETRIC EXPLANATION OF A CERTAIN OPTICAL PHENOMENON.[2]

BY WM. H. ROEVER.

Description of the Phenomenon.—In the parcel checking-room of the new Union Station at Kansas City, Missouri, there is a counter covered with brass plates which have, during the course of time, received numerous scratches by the baggage which is moved around upon the counter. The scratches are not very deep and they seem to be of fairly uniform distribution in both density and direction, as one might expect them to be after the cause of their formation has been in operation for some time. The baggage room is lighted by large electric lamps which are not very close together, so that an observer near the counter may regard the illumination in his immediate neighborhood as being due to a

[1] Steggall proves that for any regular n-gon the envelope is a point, *loc. cit.*

[2] Presented to the American Mathematical Society (Southwestern Section), December 1, 1917.

single lamp. Notwithstanding the apparently lawless nature of the manner of formation of these scratches, an observer anywhere near the counter and regardless of the direction of the illuminating electric light, will observe what appears to be a one-parameter family of illuminated ellipses which are approximately concentric and similar (see the accompanying pictures on the opposite page).

The Explanation.—A prolate spheroid, *i. e.*, an ellipsoid obtained by revolving an ellipse around its major axis, has the property that each of its points is a brilliant point with respect to its foci. In other words the focal radii to any point of the surface make equal angles with the normal to the surface at that point, and the normal lies between and in the plane of the focal radii. Consequently any reflecting surface or curve[1] which is tangent to such an ellipsoid of which a point source of light and an observer's eye are the foci, will appear to have at its point of contact with the ellipsoid a luminous point, *i. e.*, a brilliant point. The electric light and the eye of the observer are the foci of a one-parameter family of confocal ellipsoids of revolution. These ellipsoids intersect the plane of the brass-covered counter in a one-parameter family of ellipses (which are neither concentric nor similar in general, but are approximately so for the smaller curves of the family). For different positions of the observer's eye (and of the lamp, which, however, is fixed) there are, of course, different families of ellipses on the counter. Those scratches on the counter which are tangent to the members of this one-parameter family of ellipses, will have brilliant (or luminous) points at their points of contact with these curves. Owing to the fact that a scratch may have some curvature and some width and also because the source of light is not a point, it follows that not merely a point but that a small arc of the scratch becomes illuminated. These small illuminated portions of scratches (though short and disconnected) are well distributed, and even though few may lie along any individual ellipse, they do, in the aggregate, give the general impression of a one-parameter family of illuminated ellipses, *i. e.*, they make visible, so to speak, the geometric ellipses described above in much the same way that iron filings distributed in a magnetic field make visible the lines of force of that field.[2]

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[1] We will regard as a curve the exterior surface of a wire of small cross section, or the gutter-like surface of a scratch.

[2] This brilliant point phenomenon is different from any of those described by the author in the *Transactions of the American Mathematical Society*, Vol. 9, No. 2, pp. 245–279; *Bulletin of the American Mathematical Society*, Vol. XXII, No. 5, p. 218; THE AMERICAN MATHEMATICAL MONTHLY, Vol. XX, No. 10, pp. 299–303, and Vol. XXI, No. 3, pp. 69–77.